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Citation for published version:
Emmrich, E, Siska, D & Wroblewska-Kaminska, A 2016, 'Equations of second order in time with quasilinear damping: existence in Orlicz spaces via convergence of a full discretisation', *Mathematical Methods in the Applied Sciences*, vol. 39, no. 10, pp. 2449-2460. https://doi.org/10.1002/mma.3706

Digital Object Identifier (DOI):
10.1002/mma.3706

Link:
Link to publication record in Edinburgh Research Explorer

Document Version:
Peer reviewed version

Published In:
Mathematical Methods in the Applied Sciences

Equations of second order in time with quasilinear damping: existence in Orlicz spaces via convergence of a full discretisation[1]

Etienne Emmrich*

Institut für Mathematik, Technische Universität Berlin, Straße des 17. Juni 136, 10623 Berlin, Germany

David Šiška

School of Mathematics, University of Edinburgh, Peter Guthrie Tait Road, Edinburgh, EH9 3FD, UK

Aneta Wróblewska-Kamińska

Institute of Mathematics, Polish Academy of Sciences, ul. Śniadeckich 8, 00-656 Warszawa, Poland

Abstract

A nonlinear evolution equation of second order with damping is studied. The quasilinear damping term is monotone and coercive but exhibits anisotropic and nonpolynomial growth. The appropriate setting for such equations is that of monotone operators in Orlicz spaces. Global existence of solutions in the sense of distributions is shown via convergence of the backward Euler scheme combined with an internal approximation.

Keywords: Equation of second order in time with damping, monotone operator, nonpolynomial growth, anisotropic Orlicz space, existence, full discretisation, convergence
2010 MSC: 35G25, 47J35, 47H05, 65M12

1. Introduction

In this article we prove existence of solutions to a nonlinear evolution equation of second order with monotone, coercive quasilinear damping exhibiting anisotropic and nonpolynomial growth. The appropriate setting for such equations is that of monotone operators in (isotropic or anisotropic) Orlicz spaces. In general, Orlicz spaces are neither reflexive nor separable. This means that many of the usual techniques from the theory of evolution equations cannot be applied and, to our best knowledge, proof of existence of solutions in the case of an equation of

[1]This work was partially supported by Humboldt foundation and by CRC 901 *Control of self-organizing nonlinear systems: Theoretical methods and concepts of application*. In addition, AWK was partially supported by the Grant of National Science Center SONATA 5 2013/09/D/ST1/03692.

*Corresponding author

Email addresses: emmrich@math.tu-berlin.de (Etienne Emmrich), d.siska@ed.ac.uk (David Šiška), a.wroblewska@impan.pl (Aneta Wróblewska-Kamińska)

second order with quasilinear damping term with nonstandard growth is not available in the literature. Results for equations of first order in time in anisotropic Orlicz spaces can be found in [7, 11, 12], see also the references cited therein.

Let there be $a : \mathbb{R}^d \to \mathbb{R}^d$ that is continuous and monotone, i.e.,

$$(a(\xi) - a(\eta)) \cdot (\xi - \eta) \geq 0 \qquad \text{for all } \xi, \eta \in \mathbb{R}^d .$$

Moreover assume that there exists an $\mathscr{N}$-function M (see Definition 2.1 below) with its conjugate M^* and a constant $\mu \in (0, 1]$ such that

$$a(\xi) \cdot \xi \geq \mu \, (M(\xi) + M^*(a(\xi))) \qquad \text{for all } \xi \in \mathbb{R}^d , \tag{1.1}$$

where the dot denotes the Euclidean inner product. Examples can be found in [7] but some are included here for convenience.

1) $a(\xi) = \xi e^{|\xi|}$ with $M(\xi) = (|\xi| - 1)\, e^{|\xi|} + 1$, $M^*(\eta) = \left(|\xi(\eta)|^2 - |\xi(\eta)| + 1\right) e^{|\xi(\eta)|} - 1$;

2) $a(\xi) = \frac{\xi}{|\xi|} \log(|\xi| + 1) + \frac{\xi}{\log(|\xi|+1)}$ with $M(\xi) = |\xi| \log(|\xi| + 1)$, $M^*(\eta) = \frac{|\xi(\eta)|^2}{\log(|\xi(\eta)|+1)}$;

3) $a(\xi) = [|\xi_1|^{p_1-2}\xi_1, |\xi_2|^{p_2-2}\xi_2]$ $(1 < p_1,\, p_2 < \infty)$ for $\xi = [\xi_1, \xi_2] \in \mathbb{R}^2$ with $M(\xi) = \frac{1}{p_1}|\xi_1|^{p_1} + \frac{1}{p_2}|\xi_2|^{p_2}$, $M^*(\eta) = \frac{1}{q_1}|\eta_1|^{q_1} + \frac{1}{q_2}|\eta_2|^{q_2}$;

where $\xi(\eta)$ always solves the equation $\eta = a(\xi(\eta))$. Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with Lipschitz boundary $\partial\Omega$, let $[0, T]$ be a finite time interval, and let u_0, v_0, f be given problem data. We consider the existence of solutions $u : \overline{\Omega} \times [0, T] \to \mathbb{R}$ to the initial-boundary value problem

$$\partial_{tt} u - \nabla \cdot a(\nabla \partial_t u) - \Delta u = f \quad \text{in } Q := \Omega \times (0, T) , \tag{1.2a}$$

$$u = 0 \text{ on } \partial\Omega \times (0, T) , \tag{1.2b}$$

$$\partial_t u(\cdot, 0) = v_0 , \quad u(\cdot, 0) = u_0 \text{ in } \Omega. \tag{1.2c}$$

Such equations arise (though as systems), e.g., in solid mechanics describing viscoelastic material (see, e.g., [8, 17, 18], [20, p. 298], [24, pp. 928ff.]) as well as in generalizations of the Kelvin–Voigt model with nonlinear dissipation, see [2].

The coercivity assumption (1.1) later provides suitable a priori estimates for $\nabla\partial_t u$ as well as $a(\nabla\partial_t u)$, and there is no need for any extra growth condition on a whatsoever.

That Orlicz spaces are neither reflexive nor separable has already been mentioned. The main difficulty arising in this setting is that in standard functional analytic framework for the study of time-dependent partial differential equations one uses the fact that $L^p(Q)$ is isometrically isomorphic to the Bochner–Lebesgue space $L^p(0, T; L^p(\Omega))$. Thus partial differential equations are reduced to operator differential equations for functions taking values in an appropriate Banach space. This is not the case here. One cannot, in general, say that the Orlicz space over the space-time cylinder Q is isometrically isomorphic to the Orlicz space of functions defined on the time interval and taking values in the Orlicz space of functions defined on Ω.

Existence, uniqueness, and regularity of weak solutions have been studied in the literature for various types of second-order evolution equations. In the standard references [9, 20, 24] one finds, e.g., results that apply to equations of type (1.2a) if a is linearly bounded so that one can work in a Hilbert space setting. The results of the seminal paper [17] apply to the situation where a is continuous, monotone, coercive and has polynomial growth.

Convergence of discretisation methods for general classes of second-order evolution equations including (1.2a) again with polynomial growth have been considered in [4–6], see also the references cited therein, where [6] also generalises the existence results to problems with nonmonotone perturbations arising from lower order terms.

Our main result (see Theorem 4.1 below) provides global existence of a solution via convergence of a subsequence of the sequence of approximate solutions generated by a discretisation in time by the backward Euler scheme and in space by a suitable internal approximation (or Galerkin) scheme. This convergence result can also be seen as a first step towards the analysis of a practical numerical approximation employing conforming finite elements. Without assuming additional structural assumptions or regularity, however, we will not be able to improve the convergence result or to prove error estimates. We shall also remark that we were not able to prove uniqueness.

Finally, we believe that with the same techniques one may consider more general linear elliptic operators instead of $-\Delta u$ with homogeneous Dirichlet boundary conditions.

2. Notation and preliminaries

2.1. General notation

We keep the usual notation for function spaces. Let $\Omega \subset \mathbb{R}^d$ be a bounded domain. By $L^p(\Omega)$ ($p \in [1, \infty]$), we denote the Lebesgue space, for $\mathbb{R}^d$-valued functions, we write $L^p(\Omega; \mathbb{R}^d)$, both equipped with the standard norm $\|\cdot\|_{p,\Omega}$. For Sobolev spaces, we have $W^{1,p}(\Omega) = \{v \in L^p(\Omega) : \nabla v \in L^p(\Omega; \mathbb{R}^d)\}$, and $W^{1,p}_0(\Omega)$ denotes the closure of $\mathscr{C}^\infty_c(\Omega)$ with respect to the $W^{1,p}$-norm (with $H^1_0 := W^{1,2}_0$). Here, $\mathscr{C}^\infty_c(\Omega)$ denotes the space of infinitely times differentiable functions with compact support in Ω. The space of m-times uniformly continuously differentiable functions is denoted by $\mathscr{C}^m(\overline{\Omega})$ ($m \in \mathbb{N}$, $\mathscr{C} := \mathscr{C}^0$). By $\gamma_0 v$, we denote the trace of $v : \overline{\Omega} \to \mathbb{R}$ such that $\gamma_0 v = v$ on $\partial\Omega$ for smooth $v \in \mathscr{C}(\overline{\Omega})$.

With $L^p(0, T; X)$ ($p \in [1, \infty]$), we denote the usual Bochner–Lebesgue space, where X denotes a Banach space. We recall that $L^p(0, T; L^p(\Omega)) = L^p(Q)$ if $p < \infty$. Here, we identify the abstract function $u : [0, T] \to L^p(\Omega)$ with the function $u : \overline{Q} \to \mathbb{R}$ via $[u(t)](x) = u(x, t)$. The standard norm is then denoted by $\|\cdot\|_{p,Q}$. The space of functions in $L^1(0, T; X)$ whose distributional time derivative is again in $L^1(0, T; X)$ is denoted by $W^{1,1}(0, T; X)$ and equipped with the standard norm. Analogously, we define $W^{1,2}(0, T; X)$. By $\mathscr{C}([0, T]; X)$, $\mathscr{AC}([0, T]; X)$ and $\mathscr{C}_w([0, T]; X)$, we denote the usual spaces of uniformly continuous, absolutely continuous and demicontinuous (i.e., continuous with respect to the weak topology in X) functions $u : [0, T] \to X$, respectively. See also [9] for more details. By $\langle \cdot, \cdot \rangle$, we denote the duality pairing. Finally, c denotes a generic positive constant.

2.2. Orlicz spaces

In this section, we recall the definition and basic properties of Orlicz spaces (see [14] for an introduction as well as [1, 10, 13, 21, 22, 25]). Let us emphasise that our considerations include nonlinearities with anisotropic growth. We, therefore, rely upon anisotropic Orlicz classes and spaces defined by (generalised) $\mathscr{N}$-functions with vector-valued arguments (see [3, 21, 22]).

Definition 2.1 ($\mathscr{N}$-function). A continuous, convex function $M : \mathbb{R}^d \to \mathbb{R}$ is said to be an $\mathscr{N}$-function if $M(\xi) = 0$ if and only if $\xi = 0$, if $M(\xi) = M(-\xi)$ for all $\xi \in \mathbb{R}^d$, if M has superlinear growth such that $\lim_{|\xi|\to\infty} \frac{M(\xi)}{|\xi|} = \infty$, and if $\lim_{|\xi|\to 0} \frac{M(\xi)}{|\xi|} = 0$.

Note that because of the anisotropic character, the function M need not be a function that is increasing with respect to the components of its vector-valued argument.

For an $\mathscr{N}$-function M, we denote by M^* the conjugate function given by the Legendre–Fenchel transform $M^*(\eta) = \sup_{\xi \in \mathbb{R}^d} (\xi \cdot \eta - M(\xi))$ $(\eta \in \mathbb{R}^d)$. The conjugate function M^* is again an $\mathscr{N}$-function (see [21]), and $M^{**} = M$. Let us recall the Fenchel–Young inequality

$$|\xi \cdot \eta| \leq M(\xi) + M^*(\eta) \qquad \text{for all } \xi, \eta \in \mathbb{R}^d \, . \tag{2.1}$$

The *anisotropic Orlicz class* $\mathscr{L}_M(\Omega; \mathbb{R}^d)$ is the set of all (equivalence classes of almost everywhere equal) measurable functions $\xi : \Omega \to \mathbb{R}^d$ such that

$$\rho_{M,\Omega}(\xi) := \int_\Omega M(\xi(x)) \, dx < \infty \, .$$

Although $\mathscr{L}_M(\Omega; \mathbb{R}^d)$ is a convex set it may not be a linear space. The mapping $\rho_{M,\Omega}$ is a modular in the sense of [14, p. 208]. Since the function $M : \mathbb{R}^d \to \mathbb{R}$ is continuous, $\xi = \xi(x) \in L^\infty(\Omega; \mathbb{R}^d)$ implies $x \mapsto M(\xi(x)) \in L^\infty(\Omega)$, which shows that $L^\infty(\Omega; \mathbb{R}^d) \subseteq \mathscr{L}_M(\Omega; \mathbb{R}^d)$.

The *anisotropic Orlicz space* $L_M(\Omega; \mathbb{R}^d)$ is defined as the linear hull of $\mathscr{L}_M(\Omega; \mathbb{R}^d)$. It is a Banach space with respect to the Luxemburg norm

$$\|\xi\|_{M,\Omega} := \inf \left\{ \lambda > 0 : \int_\Omega M\left(\frac{\xi(x)}{\lambda}\right) dx \leq 1 \right\} ;$$

the infimum is attained if $\xi \neq 0$. Let us emphasise that, in general, $L_M(\Omega; \mathbb{R}^d)$ is neither separable nor reflexive. Note that $\rho_{M,\Omega}(\xi) \leq \|\xi\|_{M,\Omega}$ if $\|\xi\|_{M,\Omega} \leq 1$, $\rho_{M,\Omega}(\xi) \geq \|\xi\|_{M,\Omega}$ if $\|\xi\|_{M,\Omega} > 1$ for all $\xi \in L_M(\Omega; \mathbb{R}^d)$, and thus $\|\xi\|_{M,\Omega} \leq \rho_{M,\Omega}(\xi) + 1$. Finally, because of the superlinear growth of M, there holds $L_M(\Omega; \mathbb{R}^d) \subseteq L^1(\Omega; \mathbb{R}^d)$. A more detailed explanation can be found in [7]. By definition, the anisotropic Orlicz class and space coincide with the isotropic Orlicz class and space, respectively, if the $\mathscr{N}$-function $M = M(\xi)$ is a radial function.

By $E_M(\Omega; \mathbb{R}^d)$, we denote the closure with respect to the Luxemburg norm of the set of bounded measurable functions defined on Ω. Let us recall that $E_M(\Omega; \mathbb{R}^d)$ is the largest linear space contained in the Orlicz class $\mathscr{L}_M(\Omega; \mathbb{R}^d)$ so that

$$E_M(\Omega; \mathbb{R}^d) \subseteq \mathscr{L}_M(\Omega; \mathbb{R}^d) \subseteq L_M(\Omega; \mathbb{R}^d) \, ,$$

with, in general, strict inclusion. It can be shown that the Orlicz norm, given by

$$\|\xi\|^O_{M,\Omega} := \sup \left\{ \int_\Omega \xi \cdot \eta \, dx : \eta \in \mathscr{L}_{M^*}(\Omega; \mathbb{R}^d) \text{ with } \rho_{M^*,\Omega}(\eta) \leq 1 \right\} ,$$

is equivalent to the Luxemburg norm, and one finds that $L^\infty(\Omega; \mathbb{R}^d)$ is continuously embedded in $E_M(\Omega; \mathbb{R}^d)$.

The space $E_M(\Omega; \mathbb{R}^d)$ is separable and $\mathscr{C}_c^\infty(\Omega; \mathbb{R}^d)$ is dense in $E_M(\Omega; \mathbb{R}^d)$. The space $L_M(\Omega; \mathbb{R}^d)$ is the dual of $E_{M^*}(\Omega; \mathbb{R}^d)$, the duality pairing is given by

$$\langle \xi, \eta \rangle = \int_\Omega \xi \cdot \eta \, dx \, , \quad \xi \in L_M(\Omega; \mathbb{R}^d) \, , \; \eta \in E_{M^*}(\Omega; \mathbb{R}^d) \, .$$

We may recall here also the *generalised Hölder inequality*

$$\int_\Omega \xi \cdot \eta \, dx \leq 2 \, \|\xi\|_{M,\Omega} \, \|\eta\|_{M^*,\Omega} \qquad \text{for all } \xi \in L_M(\Omega; \mathbb{R}^d) \, , \; \eta \in L_{M^*}(\Omega; \mathbb{R}^d) \, ,$$

which shows that $\xi \cdot \eta \in L^1(\Omega)$ if $\xi \in L_M(\Omega;\mathbb{R}^d)$ and $\eta \in L_{M^*}(\Omega;\mathbb{R}^d)$. (The factor 2 is due to the use of the Luxemburg norm instead of the Orlicz norm.)

If the $\mathscr{N}$-function M satisfies the so-called Δ_2-*condition* (there exists $c > 0$ such that $M(2\xi) \le cM(\xi)$ for all $\xi \in \mathbb{R}^d$) then $\mathscr{L}_M(\Omega;\mathbb{R}^d) = L_M(\Omega;\mathbb{R}^d) = E_M(\Omega;\mathbb{R}^d)$ (see [1, 14, 22]). The Δ_2-condition is, however, rather restrictive. For the isotropic case, it is known that the Δ_2-condition is not fulfilled if the $\mathscr{N}$-function grows faster than a polynomial (see [14, Remark 3.4.6 on p. 138]).

In this paper, we also consider Orlicz classes and spaces over the time-space cylinder Q; the definitions and results introduced above are the same with Ω replaced by Q. We emphasise that $L_M(Q;\mathbb{R}^d) \neq L_M(0,T;L_M(\Omega;\mathbb{R}^d))$ except for the case when M is equivalent to some power function (see [3, Proposition 1.3 on p. 218]).

3. Full discretisation

In this section, we describe the numerical approximation of (1.2). For simplicity, we only consider an equidistant time grid: For $N \in \mathbb{N}$, let $\tau = T/N$ and $t_n = n\tau$ ($n = 0, 1, \ldots, N$). Besides the time discretisation, we consider a generalised internal approximation of the space

$$V := \{w \in H_0^1(\Omega) : \nabla w \in E_M(\Omega;\mathbb{R}^d)\}, \quad \|w\|_V := \|\nabla w\|_{2,\Omega} + \|\nabla w\|_{M,\Omega},$$

i.e., a sequence of (not necessarily nested) finite dimensional subspaces $V_m \subset V$ ($m \in \mathbb{N}$) with $\bigcup_{m\in\mathbb{N}} V_m$ being dense in V. In addition, we assume that $V_m \subset W^{1,\infty}(\Omega)$ ($m \in \mathbb{N}$), which is always possible. An example is given by conforming linear or bilinear finite elements (see [7, Example 3.1 on p. 1172]) if Ω is a polyhedral domain. Let $R_m : V \to V_m$ ($m \in \mathbb{N}$) denote a restriction operator such that

$$R_{m_\ell} w \to w \quad \text{in } V \text{ as } \ell \to \infty \text{ for all } w \in V \tag{3.1}$$

for any sequence $\{m_\ell\}_{\ell\in\mathbb{N}}$ with $m_\ell \to \infty$ as $\ell \to \infty$ (see also [23, pp. 25ff.]).

With respect to the right-hand side, we only consider the following restriction to the time grid: For $n = 1, 2, \ldots, N$, let $f^n := \frac{1}{\tau}\int_{t_{n-1}}^{t_n} f(\cdot,t)\,dt$.

The numerical method we consider now reads as follows: For given $u^0, v^0 \in V_m$ and $f \in L^1(0,T;L^2(\Omega))$, find $\{u^n\}_{n=1}^N, \{v^n\}_{n=1}^N \subset V_m$ such that for $n = 1, \ldots, N$

$$\int_\Omega \left(\frac{v^n - v^{n-1}}{\tau}\,\phi + a(\nabla v^n)\cdot\nabla\phi + \nabla u^n \cdot \nabla\phi \right) dx = \int_\Omega f^n \phi\,dx \quad \text{for all } \phi \in V_m, \tag{3.2a}$$

where

$$\frac{u^n - u^{n-1}}{\tau} = v^n, \tag{3.2b}$$

i.e., $u^n = u^0 + \tau\sum_{j=1}^n v^j$. The discrete solutions u^n and v^n shall approximate $u(\cdot,t_n)$ and $\partial_t u(\cdot,t_n)$, respectively.

Note that $a(\nabla v^n)$ is in $L^1(\Omega;\mathbb{R}^d)$ since $v^n \in W^{1,\infty}(\Omega)$ and $a : \mathbb{R}^d \to \mathbb{R}^d$ is continuous.

The scheme (3.2) can also be written as

$$\int_\Omega \left(\frac{u^n - 2u^{n-1} + u^{n-2}}{\tau^2}\,\phi + a\left(\nabla\frac{u^n - u^{n-1}}{\tau}\right)\cdot\nabla\phi + \nabla u^n\cdot\nabla\phi \right) dx = \int_\Omega f^n\phi\,dx \quad \text{for all } \phi \in V_m$$

for $n = 1, 2, \ldots, N$, where $u^{-1} := u^0 - \tau v^0$.

Theorem 3.1. *Let $u^0, v^0 \in V_m$ and $f \in L^1(0,T;L^2(\Omega))$ be given. Then there exists a unique solution $\{u^n\}_{n=1}^N, \{v^n\}_{n=1}^N \subset V_m$ to the numerical scheme (3.2), and the following a priori estimate is satisfied for $n = 1, 2, \dots, N$:*

$$\begin{aligned} &\|v^n\|_{2,\Omega}^2 + \sum_{j=1}^n \|v^j - v^{j-1}\|_{2,\Omega}^2 + 2\mu\tau \sum_{j=1}^n \int_\Omega \Big(M(\nabla v^j) + M^*(a(\nabla v^j)) \Big)\, dx + \|\nabla u^n\|_{2,\Omega}^2 \\ &\quad + \sum_{j=1}^n \|\nabla(u^j - u^{j-1})\|_{2,\Omega}^2 \le c \Big(\|\nabla u^0\|_{2,\Omega}^2 + \|v^0\|_{2,\Omega}^2 + \|f\|_{L^1(0,T;L^2(\Omega))}^2 \Big) . \end{aligned} \tag{3.3}$$

The proof of existence of solutions to the numerical scheme is based on the following auxiliary result, which is a direct consequence of Brouwer's fixed point theorem (see, e.g., [9, p. 74]).

Lemma 3.2. *For some $R > 0$, let $\boldsymbol{h} : \overline{B}(0,R) \to \mathbb{R}^m$ be continuous, where $\overline{B}(0,R) \subset \mathbb{R}^m$ denotes the closed ball of radius R with origin 0 with respect to some norm $\|\cdot\|_{\mathbb{R}^m}$ on $\mathbb{R}^m$. If*

$$\boldsymbol{h}(\boldsymbol{v}) \cdot \boldsymbol{v} \ge 0 \quad \textit{for all } \boldsymbol{v} \in \mathbb{R}^m \textit{ with } \|\boldsymbol{v}\|_{\mathbb{R}^m} = R$$

then there exists $\tilde{\boldsymbol{v}} \in \overline{B}(0,R)$ such that $\boldsymbol{h}(\tilde{\boldsymbol{v}}) = 0$.

Proof (of Theorem 3.1). As V_m is of finite dimension (without loss of generality, we assume that $\dim V_m = m$), we have $V_m = \operatorname{span}\{\varphi_1, \varphi_2, \dots, \varphi_m\}$ for a suitable choice of basis functions. We may then construct a one-to-one mapping between V_m and $\mathbb{R}^m$ as follows:

$$\boldsymbol{w} = [\boldsymbol{w}_1, \boldsymbol{w}_2, \dots, \boldsymbol{w}_m] \in \mathbb{R}^m \quad \leftrightsquigarrow \quad V_m \ni w = \sum_{j=1}^m \boldsymbol{w}_j \varphi_j ,$$

and $\|\boldsymbol{w}\|_{\mathbb{R}^m} := \|w\|_{2,\Omega}$ defines a norm on $\mathbb{R}^m$.

Existence and uniqueness is now shown step by step. Let us assume that u^{n-1}, $v^{n-1} \in V_m$ are given. Replacing u^n in (3.2a) by $\tau v^n + u^{n-1}$, we show that there exists $v^n \in V_m$ corresponding to $\boldsymbol{v^n} \in \mathbb{R}^m$ being a zero of the mapping $\boldsymbol{h} = [\boldsymbol{h}_1, \boldsymbol{h}_2, \dots, \boldsymbol{h}_m] : \mathbb{R}^m \to \mathbb{R}^m$ defined by

$$\boldsymbol{h}_j(\boldsymbol{w}) := \int_\Omega \left(\frac{w - v^{n-1}}{\tau} \varphi_j + a(\nabla w) \cdot \nabla \varphi_j + \tau \nabla w \cdot \nabla \varphi_j + \nabla u^{n-1} \cdot \nabla \varphi_j - f^n \varphi_j \right) dx , \quad j = 1, 2, \dots, m .$$

The continuity of $\boldsymbol{h} : \mathbb{R}^m \to \mathbb{R}^m$ is a consequence of the continuity of a together with the assumption that $V_m \subset W^{1,\infty}(\Omega)$. With the Cauchy–Schwarz inequality and the coercivity assumption (1.1), we obtain that

$$\begin{aligned} \boldsymbol{h}(\boldsymbol{w}) \cdot \boldsymbol{w} &= \int_\Omega \left(\frac{w - v^{n-1}}{\tau} w + a(\nabla w) \cdot \nabla w + \tau \nabla w \cdot \nabla w - \nabla u^{n-1} \cdot \nabla w - f^n w \right) dx \\ &\ge \frac{1}{\tau} \|w\|_{2,\Omega}^2 - \frac{1}{\tau} \|v^{n-1}\|_{2,\Omega} \|w\|_{2,\Omega} + \tau \|\nabla w\|_{2,\Omega}^2 - \|\nabla u^{n-1}\|_{2,\Omega} \|\nabla w\|_{2,\Omega} - \|f^n\|_{2,\Omega} \|w\|_{2,\Omega} \\ &= \frac{1}{\tau} \|w\|_{2,\Omega} \Big(\|w\|_{2,\Omega} - \|v^{n-1}\|_{2,\Omega} - \tau \|f^n\|_{2,\Omega} \Big) + \|\nabla w\|_{2,\Omega} \Big(\tau \|\nabla w\|_{2,\Omega} - \|\nabla u^{n-1}\|_{2,\Omega} \Big) . \end{aligned}$$

Taking $R = \|w\|_{2,\Omega}$ sufficiently large and incorporating the Poincaré–Friedrichs inequality allows us to apply Lemma 3.2 providing existence of a zero of $\boldsymbol{h}$ and thus of a solution to (3.2) at level n. Uniqueness of this solution follows immediately from the monotonicity of a together with an estimate analogous to the above one.

For deriving a priori estimates, we test (3.2a) by $\phi = v^n$ and employ again the coercivity assumption (1.1) and the Cauchy–Schwarz inequality together with the identity

$$(a-b)\cdot a = \frac{1}{2}\Big(a^2 - b^2 + (a-b)^2\Big), \tag{3.4}$$

which holds true for all $a, b \in \mathbb{R}$ as well as $a, b \in \mathbb{R}^d$. We then find

$$\frac{1}{2\tau}\Big(\|v^n\|_{2,\Omega}^2 - \|v^{n-1}\|_{2,\Omega}^2 + \|v^n - v^{n-1}\|_{2,\Omega}^2\Big) + \mu\int_\Omega \Big(M(\nabla v^n) + M^*(a(\nabla v^n))\Big)dx + \int_\Omega \nabla u^n\cdot\nabla v^n\,dx \le \|f^n\|_{2,\Omega}\|v^n\|_{2,\Omega}\,.$$

Since $v^n = (u^n - u^{n-1})/\tau$, we obtain

$$\int_\Omega \nabla u^n\cdot\nabla v^n\,dx = \frac{1}{2\tau}\Big(\|\nabla u^n\|_{2,\Omega}^2 - \|\nabla u^{n-1}\|_{2,\Omega}^2 + \|\nabla(u^n - u^{n-1})\|_{2,\Omega}^2\Big),$$

and we infer that for all $n = 1, 2, \ldots, N$

$$\begin{aligned}\|v^n\|_{2,\Omega}^2 &+ \sum_{j=1}^n \|v^j - v^{j-1}\|_{2,\Omega}^2 + 2\mu\tau\sum_{j=1}^n\int_\Omega\Big(M(\nabla v^j) + M^*(a(\nabla v^j))\Big)dx + \|\nabla u^n\|_{2,\Omega}^2\\ &+ \sum_{j=1}^n \|\nabla(u^j - u^{j-1})\|_{2,\Omega}^2 \le \|v^0\|_{2,\Omega}^2 + \|\nabla u^0\|_{2,\Omega}^2 + 2\tau\sum_{j=1}^n \|f^j\|_{2,\Omega}\|v^j\|_{2,\Omega}\,.\end{aligned} \tag{3.5}$$

Taking n such that $\|v^n\|_{2,\Omega} = \max_{j=1,2,\ldots,N}\|v^j\|_{2,\Omega} =: X$ and using that

$$\tau\sum_{j=1}^N \|f^j\|_{2,\Omega} \le \|f\|_{L^1(0,T;L^2(\Omega))}\,,$$

we come up with the quadratic inequality

$$X^2 \le \|v^0\|_{2,\Omega}^2 + \|\nabla u^0\|_{2,\Omega}^2 + 2\|f\|_{L^1(0,T;L^2(\Omega))}\,X\,.$$

This implies

$$X \le \|v^0\|_{2,\Omega} + \|\nabla u^0\|_{2,\Omega} + 2\|f\|_{L^1(0,T;L^2(\Omega))}\,.$$

Going back to (3.5), this proves the assertion. □

4. Existence via convergence of approximate solutions

In what follows, let us consider sequences $\{m_\ell\}_{\ell\in\mathbb{N}}$ and $\{N_\ell\}_{\ell\in\mathbb{N}}$ of positive integers such that $m_\ell \to \infty$ and $N_\ell \to \infty$ as $\ell \to \infty$. We do not need any coupling of the spatial and temporal discretisation parameters.

The discrete solution to (3.2) corresponding to the discretisation parameters m_ℓ, N_ℓ (with $\tau_\ell := T/N_\ell$) shall be denoted by $\{u_\ell^n\}_{n=0}^{N_\ell}$, $\{v_\ell^n\}_{n=0}^{N_\ell}$, where $u_\ell^0 \in V_{m_\ell}$ and $v_\ell^0 \in V_{m_\ell}$ denote the approximate initial values. For readability, we do not call the dependence of $t_n = n\tau_\ell$ on ℓ.

Regarding the approximation of the initial values, we assume that

$$u_\ell^0 \to u_0 \text{ in } H_0^1(\Omega) \quad \text{and} \quad v_\ell^0 \to v_0 \text{ in } L^2(\Omega) \quad \text{as } \ell \to \infty. \tag{4.1}$$

From the discrete solution, we construct approximate solutions defined on the whole time interval as follows: Let u_ℓ denote the piecewise constant function such that

$$u_\ell(\cdot,t) = u_\ell^n \quad \text{if } t \in (t_{n-1}, t_n] \ (n = 1, 2, \ldots, N_\ell), \quad u_\ell(\cdot, 0) = u_\ell^1,$$

and let $\hat{u}_\ell$ be the linear spline interpolating (t_0, u_ℓ^0), $(t_1, u_\ell^1), \ldots, (t_{N_\ell}, u_\ell^{N_\ell})$. In an analogous way, we define v_ℓ and $\hat{v}_\ell$ as well as the piecewise constant function f_ℓ.

The main result of our paper reads as follows.

Theorem 4.1. *Let $u_0 \in H_0^1(\Omega)$, $v_0 \in L^2(\Omega)$ and $f \in L^1(0,T; L^2(\Omega))$. Then there exists a solution*

$$u \in \mathscr{C}_w([0,T]; H_0^1(\Omega)) \text{ with } \partial_t u \in \mathscr{C}_w([0,T]; L^2(\Omega)),\ \nabla\partial_t u \in \mathscr{L}_M(Q;\mathbb{R}^d),\ a(\nabla\partial_t u) \in \mathscr{L}_{M^*}(Q;\mathbb{R}^d)$$

to (1.2) in the sense of distributions, i.e.,

$$\int_Q (-\partial_t u \partial_t w + a(\nabla\partial_t u)\cdot\nabla w + \nabla u \cdot \nabla w)\, dxdt = \int_Q f w dxdt \qquad \text{for all } w \in \mathscr{C}_c^\infty(Q)$$

with $u(\cdot,0) = u_0$ in $H_0^1(\Omega)$ and $\partial_t u(\cdot, 0) = v_0$ in $L^2(\Omega)$.

This solution is limit of a subsequence (denoted by ℓ') of approximate solutions constructed from (3.2) in the following sense: The piecewise constant and piecewise linear in time interpolation $u_{\ell'}$ and $\hat{u}_{\ell'}$ converge weakly in $L^\infty(0,T; H_0^1(\Omega))$ towards u; the piecewise constant and piecewise linear in time interpolation $v_{\ell'}$ and $\hat{v}_{\ell'}$ of the discrete time derivatives converge weakly* in $L^\infty(0,T;L^2(\Omega))$ towards $\partial_t u$. Moreover, $\nabla v_{\ell'}$ converges weakly* in $L_M(Q;\mathbb{R}^d)$ towards $\nabla\partial_t u$ and $a(\nabla v_{\ell'})$ converges weakly* in $L_{M^*}(Q;\mathbb{R}^d)$ towards $a(\nabla\partial_t u)$.*

Remark 4.2. Under the assumptions of Theorem 4.1, $\hat{u}_{\ell'}$ converges strongly in $\mathscr{C}([0,T]; L^2(\Omega))$ towards u. Indeed, $\{\hat{u}_\ell\} \subset \mathscr{C}([0,T]; L^2(\Omega))$ is equicontinuous since $\{\partial_t \hat{u}_\ell\}$ is bounded in $L^\infty(0,T; L^2(\Omega))$ (recall that $\partial_t \hat{u}_\ell = v_\ell$) and $\{\hat{u}_\ell(t)\} \subset H_0^1(\Omega)$ is bounded in $H_0^1(\Omega)$ and thus relatively compact in $L^2(\Omega)$ for every $t \in [0,T]$. An application of Arzelà–Ascoli's theorem thus implies strong convergence in $\mathscr{C}([0,T]; L^2(\Omega))$ of a suitably chosen subsequence $\{\hat{u}_{\ell'}\}$, and the limit can only be the weak*-limit u. It then also follows that $u_{\ell'}$ converges strongly in $L^\infty(0,T;L^2(\Omega))$ towards u since

$$\|u_\ell - \hat{u}_\ell\|_{L^\infty(0,T;L^2(\Omega))} \le \tau_\ell \, \|v_\ell\|_{L^\infty(0,T;L^2(\Omega))} \to 0 \text{ as } \ell \to \infty\,.$$

Moreover, let W be an intermediate Banach space between $L^2(\Omega)$ and $H_0^1(\Omega)$ in the sense of Lions and Peetre [16, pp. 27ff.] so that there is $c > 0$ and $\eta \in (0,1)$ such that

$$\|w\|_W \le c\, \|w\|_{H_0^1(\Omega)}^\eta \|w\|_{L^2(\Omega)}^{1-\eta} \quad \text{for all } w \in H_0^1(\Omega).$$

Then $\{\hat{u}_{\ell'}\}$ is a Cauchy sequence and thus converges strongly in $\mathscr{C}([0,T]; W)$ towards u and $u \in \mathscr{C}([0,T]; W)$. Similarly, $u_{\ell'}$ converges strongly in $L^\infty(0,T;W)$ towards u.

The proof of Theorem 4.1 requires the following approximation result.

Lemma 4.3. *Let*

$$w \in \mathscr{W} := \{w \in W^{1,1}(0,T;L^2(\Omega)) \cap L^1(0,T;H_0^1(\Omega)) : \nabla w \in \mathscr{L}_M(Q;\mathbb{R}^d)\}\,.$$

Then for any $\varepsilon > 0$ *there exists a function* $w_\varepsilon \in \mathscr{C}^1([0,T]) \otimes V$ *such that*

$$\|w_\varepsilon - w\|_{W^{1,1}(0,T;L^2(\Omega))} < \varepsilon\,, \quad \|\nabla w_\varepsilon - \nabla w\|_{L^1(0,T;L^2(\Omega;\mathbb{R}^d))} < \varepsilon\,,$$

and such that for all $\eta \in L_{M^*}(Q;\mathbb{R}^d)$

$$\Big| \int_Q \eta \cdot \nabla w_\varepsilon \, dxdt - \int_Q \eta \cdot \nabla w \, dxdt \Big| < \varepsilon\,.$$

The proof of this result follows almost the same steps as that of [7, Lemma 2.3] and essentially relies on the continuity of mollification and translation of a function in $L_M(Q;\mathbb{R}^d)$ with respect to the weak convergence in $E_M(Q;\mathbb{R}^d)$ (see [10, Lemma 1.5, 1.6] and [3, Prop. 1.2]). This leads to an approximation of $w \in \mathscr{W}$ by a smooth function vanishing at the boundary. This smooth function can then be approximated by a polynomial (with respect to the strong convergence in $\mathscr{C}^1(\overline{Q})$) and thus by an element of $\mathscr{C}^1([0,T]) \otimes V \subset \mathscr{W}$.

Proof (of Theorem 4.1). A priori estimates and convergence. In view of (4.1), the right-hand side in the a priori estimate (3.3) of Theorem 3.1 is bounded. As a consequence, we find that the sequences $\{u_\ell\}_{\ell\in\mathbb{N}}$, $\{\hat{u}_\ell\}_{\ell\in\mathbb{N}}$ are bounded in $L^\infty(0,T;H_0^1(\Omega))$ and $\{v_\ell\}_{\ell\in\mathbb{N}}$, $\{\hat{v}_\ell\}_{\ell\in\mathbb{N}}$ are bounded in $L^\infty(0,T;L^2(\Omega))$. Moreover, the sequence $\{\hat{u}_\ell(\cdot,T)\}_{\ell\in\mathbb{N}}$ (with $\hat{u}_\ell(\cdot,T) = u_\ell^{N_\ell} = u_\ell(\cdot,T)$) is bounded in $H_0^1(\Omega)$ and the sequence $\{\hat{v}_\ell(\cdot,T)\}_{\ell\in\mathbb{N}}$ is bounded in $L^2(\Omega)$. There are thus a subsequence, still denoted by ℓ, an elements $u, \hat{u} \in L^\infty(0,T;H_0^1(\Omega))$, $v, \hat{v} \in L^\infty(0,T;L^2(\Omega))$, $\xi \in H_0^1(\Omega)$, $\zeta \in L^2(\Omega)$ such that

$$\begin{aligned}
&u_\ell \overset{*}{\rightharpoonup} u\,,\ \hat{u}_\ell \overset{*}{\rightharpoonup} \hat{u} \text{ in } L^\infty(0,T;H_0^1(\Omega))\,,\ v_\ell \overset{*}{\rightharpoonup} v\,,\ \hat{v}_\ell \overset{*}{\rightharpoonup} \hat{v} \text{ in } L^\infty(0,T;L^2(\Omega))\,,\\
&\hat{u}_\ell(\cdot,T) \rightharpoonup \xi \text{ in } H_0^1(\Omega) \text{ and } \hat{v}_\ell(\cdot,T) \rightharpoonup \zeta \text{ in } L^2(\Omega)\,.
\end{aligned}$$

Since (in view of (3.3))

$$\|u_\ell - \hat{u}_\ell\|^2_{L^2(0,T,H_0^1(\Omega))} = \frac{\tau_\ell}{3} \sum_{j=1}^{N_\ell} \|\nabla(u_\ell^j - u_\ell^{j-1})\|^2_{2,\Omega} \to 0\,,$$

we find that $\hat{u} = u$. Similarly, we find that $\hat{v} = v$.

Since by definition $v_\ell = \partial_t \hat{u}_\ell$, we immediately find $v = \partial_t u$. This already shows that $u \in L^\infty(0,T;H_0^1(\Omega))$ with $\partial_t u \in L^\infty(0,T;L^2(\Omega))$ such that $u \in \mathscr{AC}([0,T];L^2(\Omega))$ and, in view of [15, Lemme 8.1 on p. 297], $u \in \mathscr{C}_w([0,T];H_0^1(\Omega))$.

Moreover by Theorem 3.1, we obtain that

$$\int_Q M(\nabla v_\ell)\, dxdt = \sum_{n=1}^{N_\ell} \int_{t_{n-1}}^{t_n} \int_\Omega M(\nabla v_\ell^n)\, dxdt \tag{4.2}$$

as well as

$$\int_Q M^*(a(\nabla v_\ell))\, dxdt = \sum_{n=1}^{N_\ell} \int_{t_{n-1}}^{t_n} \int_\Omega M^*(a(\nabla v_\ell^n))\, dxdt \tag{4.3}$$

are bounded uniformly with respect to ℓ. Thus the sequence $\{\nabla v_\ell\}_{\ell\in\mathbb{N}}$ is bounded in $L_M(Q;\mathbb{R}^d)$ and the sequence $\{a(\nabla v_\ell)\}_{\ell\in\mathbb{N}}$ is bounded in $L_{M^*}(Q;\mathbb{R}^d)$. Since $(E_M(Q;\mathbb{R}^d))^* = L_{M^*}(Q;\mathbb{R}^d)$, $(E_{M^*}(Q;\mathbb{R}^d))^* = L_M(Q;\mathbb{R}^d)$ (recall that $M^{**} = M$) and since $E_M(Q;\mathbb{R}^d)$ and $E_{M^*}(Q;\mathbb{R}^d)$ are separable normed spaces, we conclude that, passing to a subsequence if necessary,

$$\nabla v_\ell \overset{*}{\rightharpoonup} \chi \text{ in } L_M(Q;\mathbb{R}^d)\,, \quad a(\nabla v_\ell) \overset{*}{\rightharpoonup} \alpha \text{ in } L_{M^*}(Q;\mathbb{R}^d)$$

for certain χ and α. In view of (4.2) and (4.3) together with the weak sequential lower semicontinuity of the modular in $L^1(Q;\mathbb{R}^d)$ (see, e.g., [7, Lemma 2.2 on p. 1168]), we have indeed that $\chi \in \mathscr{L}_M(Q;\mathbb{R}^d)$ and $\alpha \in \mathscr{L}_{M^*}(Q;\mathbb{R}^d)$. It is obvious that $\chi = \nabla v = \nabla\partial_t u$. The limit α still has to be identified, and it remains to show in the last step of this proof that $\alpha = a(\nabla\partial_t u)$.

Using the piecewise constant and piecewise linear prolongations in time, the numerical scheme (3.2) can be rewritten as

$$\int_\Omega (\partial_t\hat{v}_\ell(\cdot,t)\phi + a(\nabla v_\ell(\cdot,t))\cdot\nabla\phi + \nabla u_\ell(\cdot,t)\cdot\nabla\phi)\,dx = \int_\Omega f_\ell(\cdot,t)\phi dx \qquad \text{for all } \phi\in V_{m_\ell}\,, \tag{4.4}$$

which holds almost everywhere as well as in the weak sense on $(0,T)$ such that

$$\begin{aligned}\int_\Omega (\hat{v}_\ell(\cdot,T)\phi\psi(T) - \hat{v}_\ell(\cdot,0)\phi\psi(0))\,dx + \int_Q (-\hat{v}_\ell\phi\psi' + a(\nabla v_\ell)\cdot\nabla\phi\psi + \nabla u_\ell\cdot\nabla\phi\psi)\,dxdt \\ = \int_Q f_\ell\phi\psi dxdt \qquad \text{for all } \phi\in V_{m_\ell}\,,\ \psi\in\mathscr{C}^1([0,T])\,.\end{aligned} \tag{4.5}$$

Taking $\phi = R_{m_\ell}w$ for arbitrary $w\in V$ and employing the weak and weak* convergence just shown together with (3.1), the strong convergence of f_ℓ in $L^1(0,T;L^2(\Omega))$ towards f (which follows from standard arguments) and the strong convergence of $\hat{v}_\ell(\cdot,0) = v_\ell^0$ in $L^2(\Omega)$ towards v_0 (by assumption), we come up with the limit equation

$$\begin{aligned}\int_\Omega (\zeta w\psi(T) - v_0 w\psi(0)\,)\,dx + \int_Q (-\partial_t u w\psi' + \alpha\cdot\nabla w\psi + \nabla u\cdot\nabla w\psi)\,dxdt = \int_Q f w\psi dxdt \\ \text{for all } w\in V\,,\ \psi\in\mathscr{C}^1([0,T])\,.\end{aligned} \tag{4.6}$$

To be precise, we have used in particular that, as $\ell\to\infty$,

$$\begin{aligned}&R_{m_\ell}w \to w \quad \text{in } L^2(\Omega)\,, \quad R_{m_\ell}w\psi' \to w\psi' \quad \text{in } L^1(0,T;L^2(\Omega))\,,\\ &\nabla R_{m_\ell}w\psi \to \nabla w\psi \quad \text{in } E_M(Q;\mathbb{R}^d)\,, \quad R_{m_\ell}w\psi \to w\psi \quad \text{in } L^1(0,T;H_0^1(\Omega))\,,\\ &R_{m_\ell}w\psi \to w\psi \quad \text{in } L^\infty(0,T;L^2(\Omega))\,.\end{aligned}$$

The above follows from (3.1) and the definition of the norm in V. Note that $\|\nabla R_{m_\ell}w\psi - \nabla w\psi\|_{M,Q} \le \max(1,T)\,\|\psi\|_{\mathscr{C}([0,T])}\,\|\nabla R_{m_\ell}w - \nabla w\|_{M,\Omega}$.

The limit equation (4.6) shows that

$$\frac{d}{dt}\int_\Omega \partial_t u w\,dx = \int_\Omega (fw - \alpha\cdot\nabla w - \nabla u\cdot\nabla w)\,dx \text{ for all } w\in V \tag{4.7}$$

in the weak sense on $(0,T)$. The right-hand side of the foregoing identity is in $L^1(0,T)$ since $f\in L^1(0,T;L^2(\Omega))$, $\alpha\in\mathscr{L}_{M^*}(Q;\mathbb{R}^d)\subseteq L^1(0,T;L_{M^*}(\Omega;\mathbb{R}^d))$ (we recall that $\|\cdot\|_{M^*,\Omega}\le\rho_{M^*,\Omega}(\cdot)+1$, see also [3, Cor. 1.1.0]), and $u \in L^\infty(0,T;H^1_0(\Omega))$. Since we already know that $\partial_t u \in L^\infty(0,T;L^2(\Omega))$, this shows that the mapping $t \mapsto \int_\Omega \partial_t u(x,t)w(x)\,dx$ is absolutely continuous on $[0,T]$ for every $w \in V$. Since V is dense in $L^2(\Omega)$ and since $\partial_t u \in L^\infty(0,T;L^2(\Omega))$, it follows that the mapping $t \mapsto \int_\Omega \partial_t u(x,t)w(x)\,dx$ is also continuous on $[0,T]$ for every $w \in L^2(\Omega)$ so that $\partial_t u \in \mathscr{C}_w([0,T];L^2(\Omega))$.

For the last step of the proof it will be crucial to know that the limit equation (4.6) does not only hold for test functions in $\mathscr{C}^1([0,T]) \otimes V$ but for a more general class of test functions. Indeed, Lemma 4.3 implies

$$\int_\Omega (\zeta w(\cdot,T) - v_0 w(\cdot,0)\,)\,dx + \int_Q (-\partial_t u \partial_t w + \alpha\cdot\nabla w + \nabla u\cdot\nabla w)\,dxdt = \int_Q fw dxdt \qquad \text{for all } w \in \mathscr{W}\,. \tag{4.8}$$

<u>Identification of initial and final values.</u> We already know that $\hat{u}_\ell$ and $v_\ell = \partial_t \hat{u}_\ell$ converges weakly* in $L^\infty(0,T;H^1_0(\Omega))$ and $L^\infty(0,T;L^2(\Omega))$ towards u and $\partial_t u$, respectively. This implies that $\hat{u}_\ell$ converges weakly in $W^{1,2}(0,T;L^2(\Omega))$ towards u. Since the trace operator Γ_0 : $W^{1,2}(0,T;L^2(\Omega)) \hookrightarrow \mathscr{C}([0,T];L^2(\Omega)) \to L^2(\Omega)$, $\Gamma_0 z := z(0)$ is linear, bounded and thus weakly-weakly continuous, we find that $\hat{u}_\ell(\cdot,0) = u^0_\ell$ converges weakly in $L^2(\Omega)$ towards $u(\cdot,0)$. On the other hand, by assumption, we know that u^0_ℓ converges strongly in $H^1_0(\Omega)$ towards u_0. This shows that $u(\cdot,0) = u_0$. Similarly, we find that $\xi = u(\cdot,T)$, where ξ was the weak in $H^1_0(\Omega)$ limit of $\hat{u}_\ell(\cdot,T) = u^{N_\ell}_\ell$.

In order to identify $\partial_t u(\cdot,0)$ and $\partial_t u(\cdot,T)$, we employ the limit equation (4.7) with the special test functions $\psi(t) = (T-t)/T$ and $\psi(t) = t/T$, respectively. So we find that for all $w \in V$

$$\begin{aligned}\frac{d}{dt}\left(\frac{T-t}{T}\int_\Omega \partial_t u(\cdot,t)w\,dx\right) &= -\frac{1}{T}\int_\Omega \partial_t u(\cdot,t)w\,dx + \frac{T-t}{T}\frac{d}{dt}\int_\Omega \partial_t u(\cdot,t)w\,dx\\ &= -\frac{1}{T}\int_\Omega \partial_t u(\cdot,t)w\,dx + \frac{T-t}{T}\int_\Omega (f(\cdot,t)w - \alpha(\cdot,t)\cdot\nabla w - \nabla u(\cdot,t)\cdot\nabla w)\,dx\,.\end{aligned}$$

We recall that the right-hand side of the foregoing identity is integrable. Integration and invoking (4.6) gives

$$-\int_\Omega \partial_t u(\cdot,0)w\,dx = -\int_\Omega v_0 w\,dx\,,$$

which shows that $\partial_t u(\cdot,0) = v_0$. Analogously, we find $\partial_t(\cdot,T) = \zeta$.

<u>Identification of the nonlinear term.</u> The main problem now arises from a lack of regularity. Indeed, the time derivative of the exact solution cannot be taken as a test function in the limit equation (4.8).

Let us start by taking $v_\ell(\cdot,t)$ as the test function in (4.4). We find

$$\int_Q a(\nabla v_\ell)\cdot\nabla v_\ell dxdt = \int_Q (f_\ell v_\ell - \partial_t \hat{v}_\ell v_\ell - \nabla u_\ell\cdot\nabla v_\ell)\,dxdt\,,$$

where (using (3.4))

$$\begin{aligned}\int_Q \partial_t \hat{v}_\ell v_\ell dxdt &= \sum_{n=1}^{N_\ell}\int_\Omega (v^n_\ell - v^{n-1}_\ell)\,v^n_\ell\,dx \geq \frac{1}{2}\sum_{n=1}^{N_\ell}\int_\Omega \left(|v^n_\ell|^2 - |v^{n-1}_\ell|^2\right)dx = \frac{1}{2}\left(\|v^{N_\ell}_\ell\|^2_{2,\Omega} - \|v^0_\ell\|^2_{2,\Omega}\right)\\ &= \frac{1}{2}\left(\|\hat{v}_\ell(\cdot,T)\|^2_{2,\Omega} - \|v^0_\ell\|^2_{2,\Omega}\right)\end{aligned}$$

and analogously

$$\int_Q \nabla u_\ell \cdot \nabla v_\ell \, dxdt = \int_Q \nabla u_\ell \cdot \nabla \partial_t \hat{u}_\ell \, dxdt \geq \frac{1}{2}\left(\|\nabla \hat{u}_\ell(\cdot, T)\|_{2,\Omega}^2 - \|\nabla u_\ell^0\|_{2,\Omega}^2\right).$$

Using $f_\ell \to f$ in $L^1(0,T;L^2(\Omega))$, $v_\ell \overset{*}{\rightharpoonup} \partial_t u$ in $L^\infty(0,T;L^2(\Omega))$, $v_\ell^0 \to v_0$ in $L^2(\Omega)$, $\hat{v}_\ell(\cdot,T) \rightharpoonup \partial_t u(\cdot,T)$ in $L^2(\Omega)$, $u_\ell^0 \to u_0$ in $H_0^1(\Omega)$, $\hat{u}_\ell(\cdot,T) \rightharpoonup u(\cdot,T)$ in $H_0^1(\Omega)$ and the weak sequential lower semicontinuity of the norm, we hence obtain

$$\limsup_{\ell\to\infty} \int_Q a(\nabla v_\ell)\cdot\nabla v_\ell dxdt \leq \int_Q f\partial_t u \, dxdt - \frac{1}{2}\left(\|\partial_t u(\cdot,T)\|_{2,\Omega}^2 - \|v_0\|_{2,\Omega}^2 + \|\nabla u(\cdot,T)\|_{2,\Omega}^2 - \|\nabla u_0\|_{2,\Omega}^2\right). \tag{4.9}$$

On the other hand, we find for arbitrary $\eta \in L^\infty(Q;\mathbb{R}^d)$ because of the monotonicity of a that

$$\begin{aligned}\int_Q a(\nabla v_\ell) \cdot \nabla v_\ell \, dxdt &\geq \int_Q a(\nabla v_\ell) \cdot \nabla v_\ell \, dxdt - \int_Q (a(\nabla v_\ell) - a(\eta)) \cdot (\nabla v_\ell - \eta) \, dxdt \\ &= \int_Q a(\nabla v_\ell) \cdot \eta \, dxdt + \int_Q a(\eta) \cdot (\nabla v_\ell - \eta) \, dxdt \, .\end{aligned}$$

Note that $a(\eta) \in E_{M^*}(Q;\mathbb{R}^d)$ since $\eta \in L^\infty(Q;\mathbb{R}^d)$ and a is continuous. In the limit, we thus obtain

$$\int_Q \alpha \cdot \eta \, dxdt + \int_Q a(\eta) \cdot (\nabla \partial_t u - \eta) \, dxdt \leq \liminf_{\ell\to\infty} \int_Q a(\nabla v_\ell) \cdot \nabla v_\ell \, dxdt \, . \tag{4.10}$$

Combining (4.9) and (4.10) yields

$$\int_Q \alpha\cdot\eta \, dxdt + \int_Q a(\eta)\cdot(\nabla\partial_t u - \eta) \, dxdt \leq \int_Q f\partial_t u \, dxdt - \frac{1}{2}\left(\|\partial_t u(\cdot,T)\|_{2,\Omega}^2 - \|v_0\|_{2,\Omega}^2 + \|\nabla u(\cdot,T)\|_{2,\Omega}^2 - \|\nabla u_0\|_{2,\Omega}^2\right) \tag{4.11}$$

still for all $\eta \in L^\infty(Q;\mathbb{R}^d)$.

We would now like to express the right-hand side of the foregoing estimate in terms of the function a. This, however, is not immediate because of the lack of regularity as mentioned earlier. We thus consider a regularisation by means of the Steklov average.

Let $h > 0$ be sufficiently small. For a function $z \in L^1(Q)$, the Steklov average is given by

$$(S_h z)(\cdot, t) = \frac{1}{2h}\int_{t-h}^{t+h} z(\cdot, s) \, ds \, , \quad t \in [0, T] \, ,$$

where z is extended by zero outside $[0, T]$.

A crucial observation now is the following: Considering the Steklov average of $\partial_t u$, we not only gain additional regularity in time but also in space. Indeed, we find that

$$S_h\partial_t u = \frac{1}{2h}\begin{cases} u(\cdot, t+h) - u(\cdot, 0) & \text{if } t \in [0, h] \, , \\ u(\cdot, t+h) - u(\cdot, t-h) & \text{if } t \in [h, T-h] \, , \\ u(\cdot, T) - u(\cdot, t-h) & \text{if } t \in [T-h, T] \, . \end{cases}$$

This shows that $S_h\partial_t u \in \mathscr{C}_w([0,T];H_0^1(\Omega))$ since $u \in \mathscr{C}_w([0,T];H_0^1(\Omega))$. Moreover, $\partial_t S_h\partial_t u = (\partial_t u(\cdot, t+h) - \partial_t u(\cdot, t-h))/(2h)$ for almost all $t \in (0,T)$ and thus $S_h\partial_t u \in W^{1,\infty}(0,T;L^2(\Omega))$ since

$\partial_t u \in L^\infty(0,T;L^2(\Omega))$. Finally, in view of Jensen's inequality, we find that $\nabla S_h \partial_t u \in \mathscr{L}_M(Q;\mathbb{R}^d)$ since $\nabla \partial_t u \in \mathscr{L}_M(Q;\mathbb{R}^d)$.

All this shows that $S_h \partial_t u \in \mathscr{W}$ is an admissible test function in (4.8), and we obtain (recalling that $\zeta = \partial_t u(\cdot,T)$)

$$\int_\Omega (\partial_t u(\cdot,T) S_h \partial_t u(\cdot,T) - v_0 S_h \partial_t u(\cdot,0)\,)\,dx + \int_Q (-\partial_t u \partial_t S_h \partial_t u + \alpha \cdot \nabla S_h \partial_t u + \nabla u \cdot \nabla S_h \partial_t u)\,dxdt = \int_Q f S_h \partial_t u\,dxdt\,. \tag{4.12}$$

Since $\partial_t u \in \mathscr{C}_w([0,T];L^2(\Omega))$ and thus the mapping $s \mapsto \int_\Omega \partial_t u(\cdot,T)\partial_t u(\cdot,s)\,dx$ is continuous on $[0,T]$, we find

$$\begin{aligned}\int_\Omega \partial_t u(\cdot,T) S_h \partial_t u(\cdot,T) dx &= \frac{1}{2h}\int_{T-h}^T \int_\Omega \partial_t u(\cdot,T)\partial_t u(\cdot,s)\,dxds \\ &\to \frac{1}{2}\int_\Omega \partial_t u(\cdot,T)^2\,dx = \frac{1}{2}\,\|\partial_t u(\cdot,T)\|_{2,\Omega}^2 \text{ as } h \to 0\,.\end{aligned} \tag{4.13}$$

Analogously, we have

$$\int_\Omega \partial_t u(\cdot,0) S_h \partial_t u(\cdot,0)\,dx \to \frac{1}{2}\,\|v_0\|_{2,\Omega}^2 \text{ as } h \to 0\,.$$

Moreover, we find that (recalling that $\partial_t u$ is extended by zero outside $[0,T]$)

$$\begin{aligned}&\int_Q \partial_t u \partial_t S_h \partial_t u\,dxdt = \frac{1}{2h}\int_0^T \int_\Omega \partial_t u(\cdot,t)\,(\partial_t u(\cdot,t+h) - \partial_t u(\cdot,t-h))\,dxdt \\ &= \frac{1}{2h}\int_0^{T-h}\int_\Omega \partial_t u(\cdot,t)\partial_t u(\cdot,t+h)\,dxdt - \frac{1}{2h}\int_h^T \int_\Omega \partial_t u(\cdot,t)\partial_t u(\cdot,t-h)\,dxdt = 0\,.\end{aligned}$$

Next, we observe that

$$\begin{aligned}&\int_Q \alpha \cdot \nabla S_h \partial_t u\,dxdt - \int_Q \alpha \cdot \nabla \partial_t u\,dxdt \\ &= \frac{1}{2h}\int_0^T \int_{t-h}^{t+h}\int_\Omega \alpha(\cdot,t) \cdot \nabla\,(\partial_t u(\cdot,s) - \partial_t u(\cdot,t))\,dxdsdt \\ &= \frac{1}{2}\int_{-1}^1 \int_0^T \int_\Omega \alpha(\cdot,t) \cdot (\nabla \partial_t u(\cdot,t+rh) - \nabla \partial_t u(\cdot,t))\,dxdtdr \to 0 \text{ as } h \to 0\end{aligned}$$

since the translation of a function in the Orlicz space $L_M(Q;\mathbb{R}^d)$ is continuous with respect to the weak convergence in $E_M(Q;\mathbb{R}^d)$ (see [10, Lemma 1.5] and [3, Prop. 1.2]).

A straightforward calculation shows that

$$\int_Q \nabla u \cdot \nabla S_h \partial_t u\,dxdt = \frac{1}{2h}\int_{T-h}^T \left(\int_\Omega \nabla u(\cdot,t) \cdot \nabla u(\cdot,T) dx\right) dt - \frac{1}{2h}\int_0^h \left(\int_\Omega \nabla u(\cdot,t) \cdot \nabla u(\cdot,0) dx\right) dt\,.$$

Since $u \in \mathscr{C}_w([0,T]; H_0^1(\Omega))$, we find similarly as before that

$$\int_Q \nabla u \cdot \nabla S_h \partial_t u \, dxdt \to \frac{1}{2}\|\nabla u(\cdot,T)\|_{2,\Omega}^2 - \frac{1}{2}\|\nabla u_0\|^2 \quad \text{as } h \to 0\,.$$

Since $S_h\partial_t u$ converges weakly* in $L^\infty(0,T; L^2(\Omega))$ towards $\partial_t u$ as $h \to 0$, we find (recalling that $f \in L^1(0,T; L^2(\Omega))$)

$$\int_Q f S_h \partial_t u \, dxdt \to \int_Q f \partial_t u \, dxdt \quad \text{as } h \to 0\,.$$

Summarising the above considerations, we obtain from (4.12)

$$\frac{1}{2}\left(\|\partial_t u(\cdot,T)\|_{2,\Omega}^2 - \|v_0\|_{2,\Omega}^2\right) + \int_Q \alpha \cdot \nabla\partial_t u \, dxdt + \frac{1}{2}\left(\|\nabla u(\cdot,T)\|_{2,\Omega}^2 - \|\nabla u_0\|_{2,\Omega}^2\right) = \int_Q f\partial_t u \, dxdt\,. \tag{4.14}$$

Using this together with (4.11) yields

$$\int_Q (\alpha - a(\eta)) \cdot (\nabla\partial_t u - \eta) \, dxdt \geq 0 \tag{4.15}$$

still for all $\eta \in L^\infty(Q; \mathbb{R}^d)$.

The remaining step is to show that $\alpha = a(\nabla\partial_t u)$. To this end, we use a variant of Minty's trick adapted to the case of nonreflexive Orlicz spaces (see also [7, 12, 19]). Let us take in (4.15)

$$\eta = \begin{cases} 0 & \text{in } Q_j \\ \nabla\partial_t u & \text{in } Q_k \setminus Q_j\,, \\ \nabla\partial_t u - \lambda\overline{\eta} & \text{in } Q \setminus Q_k\,, \end{cases}$$

where $\lambda \in (0,1)$, $\overline{\eta} \in L^\infty(Q;\mathbb{R}^d)$ and $j > k \geq 0$ are arbitrary with $Q_k = \{(x,t) \in Q : |\nabla\partial_t u(x,t)| > k\}$. Note that this choice ensures $\eta \in L^\infty(Q;\mathbb{R}^d)$. We then find

$$\int_{Q_j} (\alpha - a(0)) \cdot \nabla\partial_t u \, dxdt + \lambda \int_{Q\setminus Q_k} (\alpha - a(\nabla\partial_t u - \lambda\overline{\eta})) \cdot \overline{\eta} \, dxdt \geq 0\,.$$

The first integral on the left-hand side vanishes as $j \to \infty$ since $\alpha, a(0) \in \mathscr{L}_{M^*}(Q;\mathbb{R}^d)$, $\nabla\partial_t u \in \mathscr{L}_M(Q;\mathbb{R}^d)$ such that, with the Fenchel–Young inequality (2.1), $\alpha \cdot \nabla\partial_t u$, $a(0)\cdot\nabla\partial_t u \in L^1(Q)$ and since the measure of Q_j goes to zero as $j \to \infty$ because of $\nabla\partial_t u \in L^1(Q;\mathbb{R}^d)$.

Since a is monotone, we have

$$a(\nabla\partial_t u - \overline{\eta}) \cdot \overline{\eta} \leq a(\nabla\partial_t u - \lambda\overline{\eta}) \cdot \overline{\eta} \leq a(\nabla\partial_t u) \cdot \overline{\eta}$$

so that

$$|a(\nabla\partial_t u - \lambda\overline{\eta}) \cdot \overline{\eta}| \leq \max\left(|a(\nabla\partial_t u - \overline{\eta}) \cdot \overline{\eta}|\,, |a(\nabla\partial_t u) \cdot \overline{\eta}|\right) \in L^1(Q \setminus Q_k)\,;$$

remember here that $\nabla\partial_t u$ is bounded on $Q \setminus Q_k$. Since a is continuous, we thus find with Lebesgue's theorem on dominated convergence that

$$\int_{Q\setminus Q_k} (\alpha - a(\nabla\partial_t u - \lambda\overline{\eta})) \cdot \overline{\eta} \, dxdt \to \int_{Q\setminus Q_k} (\alpha - a(\nabla\partial_t u)) \cdot \overline{\eta} \, dxdt \text{ as } \lambda \to 0$$

and thus

$$\int_{Q\setminus Q_k} (\alpha - a(\nabla\partial_t u)) \cdot \overline{\eta}\, dxdt \geq 0\,.$$

With $\overline{\eta} = -(\alpha - a(\nabla\partial_t u))/|\alpha - a(\nabla\partial_t u)|$ if $\alpha \neq a(\nabla\partial_t u)$ and $\overline{\eta} = 0$ otherwise, we obtain

$$\int_{Q\setminus Q_k} |\alpha - a(\nabla\partial_t u)|\ dxdt = 0\,.$$

This shows that $\alpha = a(\nabla\partial_t u)$ almost everywhere in $Q \setminus Q_k$. Finally, since k is arbitrary, the equality holds almost everywhere in Q. □

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